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First-Order Saddlepoint Approximation for Reliability Analysis

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In the approximation methods of reliability analysis, nonnormal random variables are transformed into equivalent standard normal random variables. This transformation tends to increase the nonlinearity of a limit-state function and, hence, results in less accurate reliability approximation. The first-order saddlepoint approximation for reliability analysis is proposed to improve the accuracy of reliability analysis. By the approximation of a limit-state function at the most likelihood point in the original random space and employment of the accurate saddlepoint approximation, the proposed method reduces the chance of an increase in the nonlinearity of the limit-state function. This approach generates more accurate reliability approximation than the first-order reliability method without an increase in the computational effort. The effectiveness of the proposed method is demonstrated with two examples and is compared with the first- and second-order reliability methods.

Nomenclature

E	=	expectation
F	=	cumulative distribution function
f	=	probability density function
g	=	limit-state function
$\boldsymbol{H}$	=	Hessian matrix
K	=	cumulant generating function
k	=	main curvature of the limit-state function at $\boldsymbol{u}^*$
n_t	=	number of tractable random variables
$n_{\sim t}$	=	number of intractable random variables
P	=	probability
p_f	=	probability of failure
R	=	reliability
t	=	saddlepoint
U	=	standard normal random variable
$\boldsymbol{U}$	=	vector of standard normal random variables
u	=	realization of random variable U
$\boldsymbol{u}^*$	=	most probable point or most likelihood point in u space
X	=	random variable
$\boldsymbol{X}$	=	vector of random variables
x	=	realization of random variable X
$\boldsymbol{x}^*$	=	most probable point or most likelihood point in x space
Y	=	system response
β	=	reliability index
Φ	=	cumulative distribution function of standard normal distribution
Φ^{-1}	=	inverse cumulative distribution function of standard normal distribution
ϕ	=	probability density function of standard normal distribution
∇	=	gradient

I. Introduction

NUMERICAL simulations are routinely used to capture physical phenomena in detail to predict engineering system behaviors and to reduce the amount of physical testing. Because the performance and reliability of engineering systems are directly affected by the uncertainties of model parameters and model structures, it is necessary to consider uncertainties with computational simulations in the design process to ensure high reliability. Typical applications include reliability-based design[1–4] and integrated design for reliability and robustness.[5–8] Because of higher reliability requirements of an engineering system, the accuracy of the calculation of reliability or the probability of failure becomes very critical. The traditional Monte Carlo simulation[9] is generally accurate if a sufficient number of simulations are used. However, for high reliability, an excessively large number of simulations are often needed. This high computational demand is often prohibitive for complex engineering simulations such as finite element analysis and computational fluid dynamics. To overcome the shortcoming of the expensive computational cost, approximation methods have been developed[10–18] such as the first-order reliability method (FORM) and the second-order reliability method (SORM) to reduce the number of function evaluations (simulation runs). Compared to Monte Carlo simulation, both FORM and SORM are much more efficient, especially when the reliability is extremely high. Generally, SORM is more accurate than FORM, but needs more computations than FORM. In spite of its usefulness, FORM is often not accurate enough in many cases. This arouses a tradeoff consideration between the efficiency and accuracy and leads to the need for a more accurate reliability analysis method without large computational demand. To meet this need, we propose a new approximation method for reliability analysis: first-order saddlepoint approximation (FOSPA). FOSPA is generally more accurate than FORM, and in some cases more accurate than SORM, while it maintains the same order of magnitude of computational effort as FORM.

In the next section, we present the theoretical and mathematical background of this paper, including FORM, SORM, and the saddlepoint approximation. Thereafter, we present the proposed FOSPA in detail, as well as examples to demonstrate its effectiveness. The discussion and conclusion are given in more depth at the end of this paper.

II. Methods for Probability Evaluation

Essentially, the evaluation of reliability or the probability of failure by FORM and SORM is to estimate a probability, or the cumulative distribution function (CDF) of a random variable that is a function, that is, a limit-state function, of other random variables

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(basic variables), provided that the distributions of the later variables are given. Saddlepoint approximation[19] was originally developed for a related purpose, that is, to approximate the CDF of statistics of a random variable, for example, mean of random variable. In the following discussion, we will briefly review FORM, SORM, and the saddlepoint approximation. Thereafter, we will discuss the need to extend the saddlepoint approximation to reliability analysis.

A. SORM and FORM

The reliability is defined as

$$R = P\{g(\boldsymbol{X}) \geq 0\} \tag{1}$$

The probability of failure is given by

$$p_f = 1 - R = P\{g(\boldsymbol{X}) < 0\} \tag{2}$$

If the joint probability density function (PDF) of $\boldsymbol{X}$ is f_x, the probability of failure is evaluated with the integral

$$p_f = P\{g(\boldsymbol{X}) < 0\} = \int_{g(x)<0} f_x(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{3}$$

The limit-state function $g(\boldsymbol{X})$ is usually a nonlinear function of $\boldsymbol{X}$; therefore, the integration boundary is nonlinear. Because the number of random variables in practical applications is usually high, multidimensional integration is involved. Because of these complexities, there is rarely a closed-form solution to Eq. (3); it is also often difficult to evaluate the probability with numerical integration methods. When the computation cost of the limit-state function is relatively cheap, Monte Carlo integration is often applied to the problem. However, when Monte Carlo simulation is not computationally affordable, approximation methods such as FORM[10] and SORM[11] have become the methods of choice in practical applications. These approximation methods involve the following steps: 1) transformation of random variables from their original random space into a standard normal space, 2) optimization process to find the most probable point (MPP): the design point with the highest contribution to the integral calculation in Eq. (3), 3) linear (in FORM) or quadratic approximation (in SORM) of the limit-state function in the standard normal space at the MPP, and 4) calculation of probability by the use of normal distribution tail approximation.

In the first step, the original random variables $\boldsymbol{X} = \{X_1, X_2, \ldots, X_n\}$ (in x space) are transformed into a set of random variables $\boldsymbol{U} = \{U_1, U_2, \ldots, U_n\}$ (in u space) whose elements follow a standard normal distribution. The transformation is given by[20]

$$u_i = \Phi^{-1}\left\{F_{x_i}(x_i)\right\} \tag{4}$$

The probability integration is then rewritten as

$$P\{g(\boldsymbol{X}) < 0\} = \int_{g(u)<0} f_u(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \tag{5}$$

Note that after the transformation, the integration in Eq. (5) in u space is identical to the integration in Eq. (3) in x space without any loss of accuracy, and the contours of the integrand $f_u(\boldsymbol{u})$ become concentric hyperspheres. The motivation for the use of the transformation formulation in Eq. (5) instead of Eq. (3) to calculate the probability of failure will become clear in the following discussion.

To make the integration calculation in Eq. (5) easier, in addition to making the integrand more regular (concentric hypercircle contours), the integral boundary $g(\boldsymbol{u})$ is also approximated linearly with the first-order Taylor expansion as

$$g(\boldsymbol{U}) \approx g(\boldsymbol{u}^*) + \nabla(\boldsymbol{u}^*)(\boldsymbol{U} - \boldsymbol{u}^*) \tag{6}$$

or with the second-order Taylor expansion as

$$g(\boldsymbol{U}) \approx g(\boldsymbol{u}^*) + \nabla(\boldsymbol{u}^*)(\boldsymbol{U} - \boldsymbol{u}^*) + \tfrac{1}{2}(\boldsymbol{U} - \boldsymbol{u}^*)^T \boldsymbol{H}(\boldsymbol{u}^*)(\boldsymbol{U} - \boldsymbol{u}^*) \tag{7}$$

where $\boldsymbol{u}^*$ is the expansion point. Equation (6) is used in FORM and Eq. (7) is used in SORM.

To reduce the loss of accuracy to a minimum degree, it is natural to expand the function $g(\boldsymbol{U})$ at a point that has the highest contribution to the probability integration. Therefore, the MPP is considered as the expansion point. The MPP is the point on the surface of $g(\boldsymbol{U}) = 0$ for which the PDF of $\boldsymbol{U}$ is at its maximum. Maximizing the joint PDF of $\boldsymbol{U}$ on the surface of $g(\boldsymbol{U}) = 0$ and noting that the contour $f_u(\boldsymbol{u})$ is a concentric hypersphere, we have the following formulation for locating the MPP:

$$\min_u \|\boldsymbol{u}\| \qquad \text{subject to} \qquad g(\boldsymbol{u}) = 0 \tag{8}$$

where $\| \; \|$ stands for the norm (length) of a vector.

Geometrically, the MPP is the shortest distance point from surface $g(\boldsymbol{u}) = 0$ to the origin in u space, and the minimum distance $\beta = \|\boldsymbol{u}^*\|$ is called reliability index. From Eqs. (5) and (6), the probability of failure is approximated by FORM as

$$p_f = P\{g(\boldsymbol{X}) < 0\} = \Phi(-\beta) \tag{9}$$

From Eqs. (5) and (7), SORM[11] gives the following approximation,

$$p_f = P\{g(\boldsymbol{X}) < 0\} = \Phi(-\beta) \prod_{i=1}^{n-1} (1 + \beta\kappa_i)^{\frac{1}{2}} \tag{10}$$

Generally, because the approximation of the limit-state in SORM [Eq. (7)] is better than that in FORM, the accuracy of SORM is higher than that of FORM [Eq. (6)].

B. Saddlepoint Approximation

Daniels[19] introduced the saddlepoint approximation technique for the approximation distribution of statistics, for example, the mean, by integration of its density estimate. Since Daniels's work, especially after 1980, research and applications in this area have vastly increased.[21–30] Instead of direct approximation of the probability integration in Eq. (2), saddlepoint approximation uses a Fourier inversion formula (in an integral form) to approximate a PDF. Let Y be a random variable distributed according to the density function $f(y)$. The moment generating function of Y is defined as

$$M(\xi) = \int_{-\infty}^{+\infty} e^{\xi y} f(y)\,\mathrm{d}y \tag{11}$$

and the cumulant generating function (CGF) of Y is defined as

$$K(\xi) = \log\{M(\xi)\} \tag{12}$$

To restore $f(y)$ from $K(\xi)$, we can apply the inverse Fourier formula

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} M(i\xi)e^{-i\xi y}\,\mathrm{d}\xi$$

$$= \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} \exp\{K(\xi) - \xi y\}\,\mathrm{d}\xi \tag{13}$$

Using exponential power series expansions to evaluate the integral in Eq. (13) and the Hermite polynomials approximation, Daniels[19] arrived to the so-called saddlepoint approximation to $f(y)$ as

$$f(y) = \{1/2\pi K''(t)\}^{\frac{1}{2}} \exp\{K(t) - ty\} \tag{14}$$

where $K''(t)$ is the second derivative of the CGF with respect to t, where t is the saddlepoint corresponding to the solution to the following equation:

$$K'(t) = y \tag{15}$$

The central idea of deriviation of Eq. (14) is to choose the integral path passing through the saddlepoint of the integrand, where the integrand is approximated. Because the saddlepoint is an extreme point, the function of integrand falls away rapidly as we move from this point. Thus, the influence of neighboring points on the integral in Eq. (13) is diminished.[29] Interested readers should consult Goutis and Casella[29] for a good explanation of saddlepoint approximation. For the comprehensive methodology, one can refer to Ref. 30.

Although the theory of saddlepoint approximation is quite complex, its use, especially the CDF approximation version, is fairly straightforward.[21] The approximation of CDF of Y by the saddlepoint approximation derived by Lugannani and Rice[23] is

$$F_Y = P\{Y \le y\} = \Phi(w) + \phi(w)(1/w - 1/v) \tag{16}$$

or, alternatively by Barndorff-Nielsen,[24]

$$F_Y = P\{Y \le y\} = \Phi\{w + (1/w)\log(v/w)\} \tag{17}$$

where

$$w = \operatorname{sign}(t)\{2[ty - K(t)]\}^{\frac{1}{2}} \tag{18}$$

$$v = t\{K''(t)\}^{\frac{1}{2}} \tag{19}$$

in which $\operatorname{sign}(t) = +1, -1$, or 0, depending on whether t is positive, negative, or zero.

Daniels[19] discusses the existence and properties of the real roots to Eq. (15), on which the saddlepoint approximation depends, and concludes that the saddlepoint approximation can be used whenever t lies with the restricted range assumed by $K'(t)$ where Eq. (15) has a unique real root.

From Eqs. (16) and (17), we see that the CDF of Y is approximated by the use of standard normal distribution, as shown by the use of CDF and PDF of the standard normal distribution in Eqs. (16) and (17). Wood et al.[27] derived a general saddlepoint formula where the normal-base distribution is replaced by a general-base distribution.

As indicated by many previous researchers, for example, in Ref. 19, the saddlepoint approximation yields extremely good accuracy for CDF, especially for the tail area of a distribution, whereas it requires only the process of finding one saddlepoint without any integration. In terms of accuracy and efficiency, there is a great potential to extend this technique to reliability analysis and eventually to probabilistic engineering design.

Because the saddlepoint approximation method involves the CGF and its derivatives, the major requirement for applications of the technique is the tractability, that is, the existence of a CGF of the distribution of random variable Y. For an engineering application, Y is a system performance, that is, limit-state function, that is dependent on basic random variables $\boldsymbol{X}$, that is, $Y = g(\boldsymbol{X})$. The key to application of the saddlepoint approximation to a general performance Y is to find the CGF of Y provided that distributions of $\boldsymbol{X}$ are given. In this papers, a general FOSPA method is developed with the capability to evaluate the CDF of a limit-state function accurately for any continuous distributions of basic variables.

III. FOSPA Reliability Method

The calculation error of the probability of failure of FORM comes from the linear approximation [Eq. (6)] to the limit-state state function in u space. The error of SORM comes from two sources, one is the quadratic approximation [Eq. (7)] to the limit-state function in u space and the other is the approximation of probability integration for the approximated limit-state function in the quadratic form. For a detailed discussion on the error of FORM and SORM, refer to Ref. 31. Even though FORM gives an accurate solution to the probability integration for the approximated limit-state function (a linear function), it is generally less accurate than SORM because of the linear approximation. The fact that SORM is generally more accurate than FORM implies that the accuracy of the limit-state function approximation is very important to ensure a highly accurate reliability estimation.

Though the nonnormal to normal transformation makes it possible and easy to calculate the probability of failure analytically (without simulations), the transformation generally increases the nonlinearity of a limit-state function because the transformation in Eq. (4) is nonlinear. For example, if a limit state is a linear function of nonnormal random variables, after the transformation by the use of Eq. (4), it will become a nonlinear function of standard normal random variables. If the approximation to the limit-state function at the MPP in u space cannot capture the nonlinearity well, the accuracy of the probability approximation will become unacceptable. To reduce the accuracy loss to the minimum extent, we need to avoid or reduce the chance of an increase in the nonlinearity due to the transformation of random variables. In other words, we may consider approximating a limit-state function in the original x space or avoid unnecessary transformation as much as possible.

To address the aforementioned concerns, we propose the FOSPA to improve the accuracy of reliability analysis while maintaining the same efficiency as FORM. In FOSPA, the limit-state function is linearized in the original random space at the so-called most likelihood point (MLP). If all of the random variables are tractable, then the saddlepoint approximation can be directly applied. If some of the random variables do not have CGF, they are transformed into other random variables that have CGF before the linearization. In the following, we will discuss FOSPA in three cases: 1) all of the random variables are tractable, 2) some of the random variables are tractable, and 3) none of the random variables is tractable. Strictly speaking, by tractable we mean that a random variable has a closed-form of CGF; otherwise, we call the random variable intractable. At the end of this section, we will present a general procedure and computational aspect of FOSPA implementation.

A. Case 1: All of the Random Variables Are Tractable

The limit-state function $g(\boldsymbol{X})$ is first linearized at some point $\boldsymbol{x}^*$, namely, the integral boundary of Eq. (3) is approximated by a hyperplane at $\boldsymbol{x}^*$. Similar to the concept of the MPP, the expansion point $\boldsymbol{x}^*$ is chosen such that the joint PDF of $\boldsymbol{X}$ is at its maximum value on the boundary of the limit state $g(\boldsymbol{X}) = 0$; this point is called the MLP. In other words, the MLP is the point on the boundary $g(\boldsymbol{X}) = 0$, that has the highest contribution to the probability of failure:

$$p_f = \int_{g(x)<0} f_x(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

The following model is used to identify the MLP $\boldsymbol{x}^*$:

$$\max_x \prod_{i=1}^{n} f_i(x_i)$$
$$\text{subject to} \qquad g(\boldsymbol{x}) = 0 \tag{20}$$

The linear form of $g(\boldsymbol{X})$ at $\boldsymbol{x}^*$ is

$$g(\boldsymbol{X}) \approx \nabla(\boldsymbol{x}^*)(\boldsymbol{X} - \boldsymbol{x}^*) \tag{21}$$

Then the CGF of $g(\boldsymbol{X})$ is given by

$$K(t) = \sum_{i=1}^{n} K_i(t) \tag{22}$$

where $K_i(t)$ is the CGF of $\nabla_i(\boldsymbol{x}^*)(X_i - x_i^*)$.

The first and second derivatives of $K(t)$ are

$$K'(t) = \sum_{i=1}^{n} K_i'(t) \tag{23}$$

$$K''(t) = \sum_{i=1}^{n} K_i''(t) \tag{24}$$

respectively.

Table 1 CGF of some distributions

Distribution	PDF	CGF
Normal	$f(x)=(1/\sqrt{2\pi}\sigma)\exp[(x-\mu)^2/2\sigma^2]$	$K(t)=\mu t+\frac{1}{2}\sigma^2t^2$
Exponential	$f(x)=\beta\exp(-\beta x)$	$K(t)=-\ln(1-t/\beta)$
Uniform	$f(x)=1/(b-a)$	$K(t)=\ln[\exp(bt)-\exp(at)]-\ln(b-a)-\ln(t)$
Gumbel	$f(x)=(1/\sigma)\exp[(x-\mu)/\sigma]\exp\{-\exp[(x-\mu)/\sigma]\}$	$K(t)=\mu t+\log\Gamma(1-\sigma t)$
χ^2	$f(x)=[1/\Gamma(n/2)2^{n/2}]x^{n/2-1}\exp(-\frac{1}{2}x)$	$K(t)=-\frac{1}{2}n\ln(1-2t)$
Gamma	$f(x)=[\beta^\alpha/\Gamma(\alpha)]x^{\alpha-1}\exp(-\beta x)$	$K(t)=\alpha\{\ln(\beta)-n(\beta-t)\}$

According to Eq. (15), the saddlepoint t is identified by the solution to the equation

$$K'(t)-y=\sum_{i=1}^{n}K_i'(t)-y=0 \tag{25}$$

Once the saddlepoint t is identified, the probability $P\{g(\boldsymbol{X})\le y\}$ can be calculated from Eq. (16) with the equations

$$w=\mathrm{sign}(t)\{2[ty-K(t)]\}^{\frac{1}{2}}=\mathrm{sign}(t)\left(2\left[ty-\sum_{i=1}^{n}K_i(t)\right]\right)^{\frac{1}{2}} \tag{26}$$

$$v=t\{K''(t)\}^{\frac{1}{2}}=t\left\{\sum_{i=1}^{n}K_i''(t)\right\}^{\frac{1}{2}} \tag{27}$$

The CGFs of some common distributions are listed in Table 1. For further details, refer to Ref. 32.

B. Case 2: Some of the Random Variables Are Tractable

Some random variables may not have a closed form that is, intractable, CGF, for example, Weibull distribution and lognormal distribution. There are two ways to approach intractable CGF: 1) approximate the CGF using polynomial expansions[33] or 2) transform the random variable into another random variable with tractable CGF. The latter approach is adopted in this paper for the purpose of simplicity. One possible transformation is similar to the one used in FORM and SORM as shown in Eq. (4), which is the transformation from a random variable with intractable CGF to a standard normal variable. In general, any distribution with tractable CGF can be used for the transformation.

Let the set of variables that have tractable CGF be $\boldsymbol{X}^t=\{X_i^t;\ i=1,2,\ldots,n_t\}$ and the set of variables without tractable CGF be $\boldsymbol{X}^{\sim t}=\{X_j^{\sim t};\ j=1,2,\ldots,n_{\sim t}\}$. After the nonnormal–normal transformation, $\boldsymbol{X}^{\sim t}$ is transformed into a set of standard normal variables $\boldsymbol{U}=\{U_j;\ j=1,2,\ldots,n_{\sim t}\}$. Then, the formulation for searching the MLP $\{\boldsymbol{x}_t^*,\boldsymbol{u}^*\}$ becomes

$$\max_{x^t,u}\prod_{i=1}^{n_t}f_i\left(x_i^t\right)\prod_{j=1}^{n_{\sim t}}\phi(u_j)$$
$$\text{subject to}\qquad g(\boldsymbol{x}^t,\boldsymbol{u})=0 \tag{28}$$

After linearization, the limit-state function at the MLP $\{\boldsymbol{x}^{t^*},\boldsymbol{u}^*\}$ is given by

$$g(\boldsymbol{X})\approx q(\boldsymbol{X}^t,\boldsymbol{U})=\sum_{i=1}^{n_t}\left.\frac{\partial g}{\partial x_i^t}\right|_{x^{t*},u^*}\left(X_i^t-x_i^{t^*}\right)+\sum_{j=1}^{n_{\sim t}}\left.\frac{\partial g}{\partial u_j}\right|_{x^{t*},u^*}\left(U_j-u_j^*\right) \tag{29}$$

Because the limit state in Eq. (29) is a linear combination of tractable random variables, the saddlepoint approximation method in Eqs. (22–27) can be applied in conjunction with Eq. (29) to evaluate the probability of failure.

C. Case 3: None of the Random Variables Is Tractable

When all random variables are intractable, they must be transformed into selected tractable random variables such as standard normal variables. If all of the random variables are transformed into standard normal variables, after the transformation, the model of searching the MLP becomes

$$\max_{u}\prod_{i=1}^{n}\phi(u_i)$$
$$\text{subject to}\qquad g(\boldsymbol{u})=0 \tag{30}$$

which is equivalent to the model in Eq. (8) for the MPP search. Therefore, the solution $\boldsymbol{u}^*$ to the model in Eq. (30) is exactly the MPP defined in the model (8). At the MLP, the linearization of the limit-state function is given by

$$g(\boldsymbol{X})=\sum_{i=1}^{n}\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\left(U_i-u_i^*\right) \tag{31}$$

Appendix A shows that the calculated probability of failure from saddlepoint approximation based on Eq. (31) is the same result as that of FORM. In other words, FORM is identical to FOSPA when all random variables are transformed into standard normal variables. Therefore, FORM is a special case of FOSPA.

D. General Procedure and Computation Implementation of FOSPA

The procedure of FOSPA is summarized as follows:

1) Determine whether a random variable has tractable or intractable CGF and form two sets of random variables, one set with tractable CGF, $\boldsymbol{X}^t$, and the other set without tractable CGF, $\boldsymbol{X}^{\sim t}$. Transform the latter set into standard normal variables $\boldsymbol{U}$.

2) Solve the model in Eq. (28) to identify the MLP $\{\boldsymbol{x}^{t^*},\boldsymbol{u}^*\}$.

3) Linearize the limit-state function at the MLP as shown in Eq. (29).

4) Formulate the saddlepoint equation and solve it to obtain the saddlepoint t.

5) Use Eqs. (15–19) to find the probability of failure.

Note that if all of the random variables are tractable, $\boldsymbol{X}^{\sim t}$ will be an empty set and the problem belongs to case 1, and if none of the random variables is tractable, $\boldsymbol{X}^t$ will be an empty set and the problem belongs to case 3 where the same result as that of FORM will be obtained.

To make the numerical computation process of FOSPA more stable, several practical measures may be considered, and some of them are briefly discussed here. The variables in Eqs. (20) and (28) for MLP search are normalized by the means and standard deviations of the random variables. This normalization sets the design variables in the same scales. Note that this normalization is a linear transformation and will not affect the nonlinearity of the limit-state function, but it will help the convergence of the iterative process of finding the MLP. To avoid the objective functions of the MLP search in Eqs. (20) and (28) becoming too small, one may choose to use the natural logarithm of the objective functions. To avoid singularities in Eqs. (18) and (19), one may use the reverse sign of the limit-sate function when a square root of a negative value occurs.

Given that there is a strong need to minimize the number of limit-state function evaluations, so that the technique is practical for computationally expensive engineering simulation models, for example, finite element analysis and computational fluid dynamics, we compare the efficiencies of the methods by counting the number of function evaluations of limit-state function. Because FOSPA uses a similar optimization formulation to find the MLP as FORM does for the MPP, and because it uses a less nonlinear constraint function, the computational effort (measured by the number of function evaluations) of FOSPA is less than or at most the same as that of FORM.

IV. Numerical Examples

In this section, two examples are used to demonstrate the effectiveness of the proposed method. The first example is associated with a linear limit-state function and the other with a nonlinear limit-state function. We will compare the accuracy and efficiency among FOSPA, FORM, and SORM. If no theoretical solution exists, we will use the result of Monte Carlo simulation with a relatively large sample size as a reference. In the following examples, the first- and second-order derivatives are evaluated numerically with the finite difference method. Because of this finite difference calculation, SORM, which requires second-order derivative information, has an inherent inefficiency in terms of the number of function evaluations.

A. Example 1: Linear Limit-State Function

A linear limit-state function is given by a sum of independent random variables,[11] as follows:

$$g(\boldsymbol{X}) = \left(n + a\sqrt{n}\right) - \sum_{i=1}^{n} X_i \tag{32}$$

where a is a constant and X_i are n independent random variables.

1. Case 1: All Random Variables Are Tractable

Let each of the random variables follows a standard exponential distribution with CDF

$$F(x_i) = 1 - \exp(-x_i) \tag{33}$$

For this specific example, the theoretical solution can be found. The probability of failure $p_f = P\{g(\boldsymbol{X}) < 0\}$ is listed in Table 2 and depicted in Fig. 1 for $n = 2$.

Figure 1 shows the probability of failure for different values of a. The probability of failure changes in the range roughly between 0.4 and 0 as a varies. The curves of FOSPA and the exact solution almost overlap each other over the whole range of the probability. This indicates that FOSPA is consistently good over the range of probability of failure. SORM is more accurate than FORM, but when the probability of failure is high, for example, 0.4, SORM is not accurate, as shown in Fig. 1. The accuracy of solution from SORM increases as the probability of failure becomes lower. This phenomenon conforms to the fact that SORM is only accurate at the tail of a distribution due to its asymptotic approximation to the probability integration.[11] In this example, with linear limit-state function and tractable CGF random variables, the results show that FOSPA is the most accurate method.

Figure 2 shows that when $a = 3.5$, the original linear limit-state function becomes highly nonlinear after the transformation to standard normal distributions required by both FORM and SORM. The linear approximation of FORM is far away from the transformed

Table 2 Probability $p_f = P\{g(X) < 0\}$ for $n = 2$

a	FORM	SORM	FOSPA	Exact
0.0001	0.3166	0.3612	0.4068	0.4060
0.5	0.1795	0.2301	0.2482	0.2474
1.0	0.0990	0.1393	0.1459	0.1452
1.5	0.0536	0.0816	0.0835	0.0831
2.0	0.0286	0.0466	0.0469	0.0466
2.5	0.0152	0.0262	0.0260	0.0258
3.0	0.0079	0.0145	0.0142	0.0141
3.5	0.0041	0.0079	0.0077	0.0076
4.0	0.0021	0.0043	0.0041	0.0041
4.5	0.0011	0.0023	0.0022	0.0022
5.0	0.0006	0.0012	0.0012	0.0012
5.5	0.0003	0.0007	0.0006	0.0006
6.0	0.0001	0.0003	0.0003	0.0003

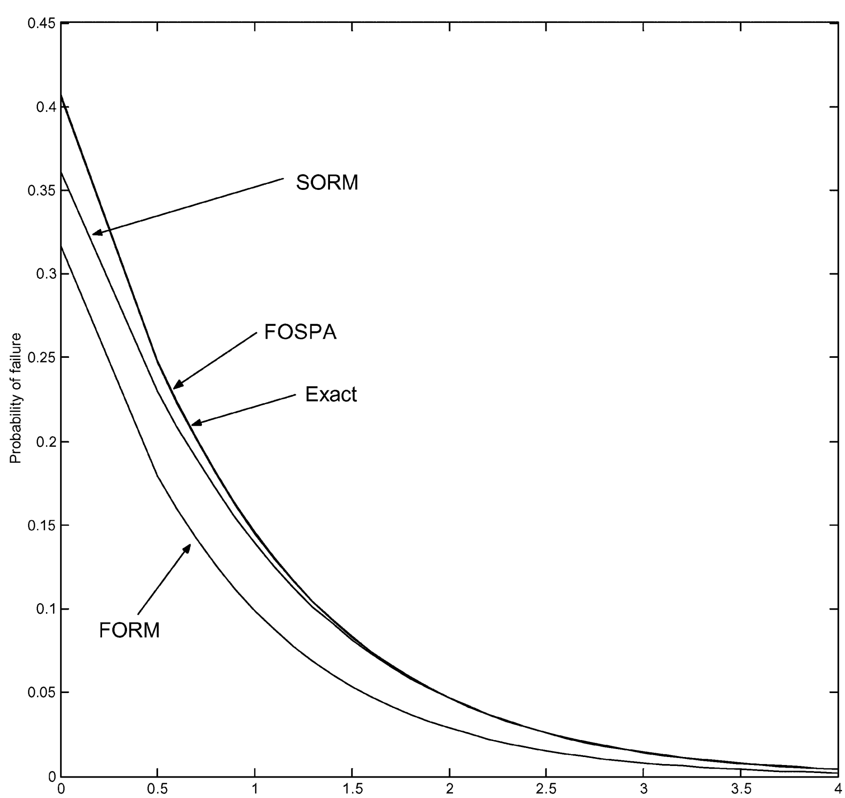

Fig. 1 Probability of failure when $n = 2$.

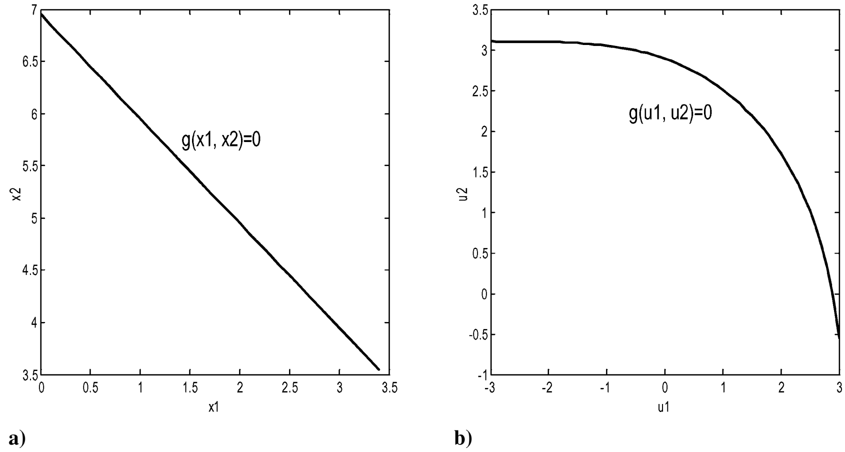

Fig. 2 Limit-state function in a) *x* and b) *u* spaces.

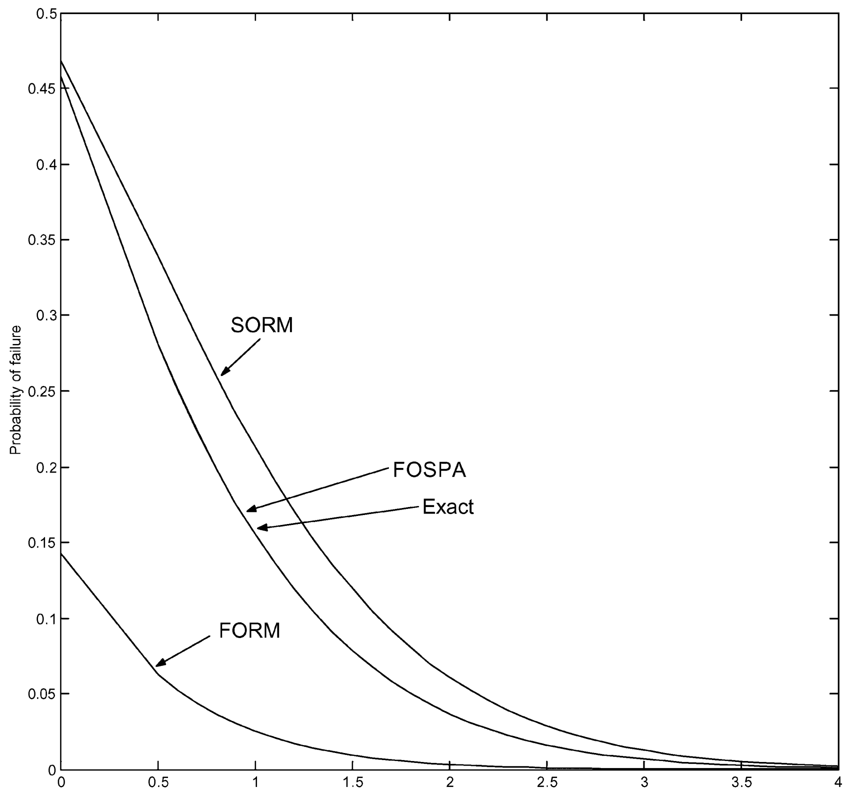

Fig. 3 Probability of failure when *n* = 10.

nonlinear limit-state function in u space, and even the quadratic approximation in SORM cannot capture the nonlinearity of the transformed limit-state function very well. Therefore, both FORM and SORM are not as accurate as FOSPA in this example. Because FOSPA uses the original linear limit-state function without the increase of nonlinearity and saddlepoint approximation, a high-accuracy approximation results. That is, the overall accuracy of FOSPA is superior to FORM and SORM.

The result for higher dimension with $n = 10$ is listed in Table 3 and shown in Fig. 3. The result still shows that the FOSPA is much more accurate than FORM and SORM. The related detailed equations used in this example are given in Appendix B.

2. Case 2: Some Random Variables Are Not Tractable

In the following case, we choose X_3 to follow a Weibull distribution that does not have a closed-form CGF. The distribution information is shown in Table 4.

Because X_3 is not tractable, it is transformed into a standard normal variable before the saddlepoint approximation is applied. Monte Carlo simulation (MCS) is employed, and its result is used as a reference for comparison of the accuracy of other methods. The number of simulations in the MCS is 10^6. The calculated probability of failure is shown in Table 5. Note that FOSPA is the most accurate method and that SORM is more accurate than FORM. With FORM and SORM, the transformation of $\{X_1, X_2, X_3\}$ into a standard normal variable $\{U_1, U_2, U_3\}$ makes the original linear limit-state function become nonlinear in terms of $\{U_1, U_2, U_3\}$. On the other hand, FOSPA only involves the transformation of X_3 into a standard normal variable U_3. That is, the original limit state is only nonlinear in terms of U_3, and the remaining terms of X_1 and X_2 are kept linear. As a result of the minimum increase of nonlinearity of the limit state, FOSPA is more accurate than FORM and SORM. The numbers of function evaluations used by FOSPA, FORM, and SORM (including finite difference calculation and iterations to find MLP/MPP) are 25, 37, and 57, respectively. In this case, the minimum increase of nonlinearity also helps FOSPA to be the most efficient method for finding the MLP, whereas SORM is the least efficient method for this specific case.

Table 3 Probability $p_f = P\{g(X) < 0\}$ for $n = 10$

a	FORM	SORM	FOSPA	Exact
0.0001	0.1429	0.4683	0.4580	0.4579
0.5000	0.0628	0.3392	0.2810	0.2809
1.0000	0.0253	0.2131	0.1554	0.1554
1.5000	0.0094	0.1195	0.0786	0.0786
2.0000	0.0033	0.0610	0.0369	0.0369
2.5000	0.0011	0.0288	0.0162	0.0162
3.0000	0.0004	0.0127	0.0067	0.0067
3.5000	0.0001	0.0053	0.0027	0.0027
4.0000	0.0000	0.0021	0.0010	0.0010
4.5000	0.0000	0.0008	0.0004	0.0004
5.0000	2.85*e*–6	2.96*e*–4	1.29*e*–4	1.29*e*–4
5.5000	8.03*e*–7	1.05*e*–4	4.43*e*–5	4.44*e*–5
6.0000	2.22*e*–7	3.63*e*–5	1.47*e*–5	1.47*e*–5

Table 4 Information of random variables

Variable	Parameter 1	Parameter 2	Distribution
X_1	1.2	——	Exponential[a]
X_2	1.2	——	Exponential
X_3	2	1.5	Weibull[b]

[a]For an exponential distribution, parameter 1 is the mean.
[b]For a Weibull distribution, parameters 1 and 2 are parameters a and b, respectively, in the PDF of a Weibull distribution $f(x) = abx^{b-1}e^{-ax^b}$.

Table 5 Probability of failure for case 2

Failure probability	a	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X}) < 0\}$	4.3	1.224×10^{-4}	3.025×10^{-4}	2.770×10^{-4}	2.289×10^{-4}
N[a]	——	25	37	25	10^6

[a]Number of function evaluations.

Table 6 Information of random variables

Variable	Parameter 1	Parameter 2	Distribution
X_1	2	1.5	Weibull
X_2	2	1.5	Weibull
X_3	2	1.5	Weibull

Table 7 Probability $P\{g(X) < 0\}$ for case 3

Failure probability	a	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X}) < 0\}$	0.5	0.0026	0.0038	0.0026	0.0040
N		21	27	21	10^6

3. Case 3: All of the Random Variables Are Not Tractable

In the following case, all random variables follow Weibull distributions as shown in Table 6. Because a Weibull distribution does not have tractable CGF, the transformation from $\{X_1, X_2, X_3\}$ to a standard normal variable $\{U_1, U_2, U_3\}$ is required.

As expected, FOSPA has the same result as FORM as shown in Table 7. The MLP from FOSPA and the MPP from the FORM are identical, that is, $\{x_1^{\mathrm{MLP}}, x_2^{\mathrm{MLP}}, x_3^{\mathrm{MLP}}\} = \{x_1^{\mathrm{MPP}}, x_2^{\mathrm{MPP}}, x_3^{\mathrm{MPP}}\} = \{1.2887, 1.2887, 1.2887\}$. In this case, SORM is the most accurate method because the second-order approximation in SORM provides a better approximation to the limit-state function in u space.

B. Example 2: Nonlinear Limit-State Function

Consider the limit-state function of a shaft in a speed reducer defined as

$$g(\mathbf{X}) = S - (32/\pi D^3)\sqrt{F^2L^2/16 + T^2} \tag{34}$$

Table 8 Distributions of random variables

Variable	Parameter 1	Parameter 2	Distribution
Diameter D	39 mm	0.1 mm	Normal[a]
Span L	400 mm	0.1 mm	Normal
External force F	1500 N	350 N	Gumbel[b]
Torque T	250 Nm	35 Nm	Normal
Strength S	70 MPa	80 MPa	Uniform[c]

[a]For normal distribution, parameters 1 and 2 are mean and standard deviation, respectively.
[b]For Gumbel distribution, parameters 1 and 2 are mean and standard deviation, respectively.
[c]For a uniform distribution, parameters 1 and 2 are lower and upper bounds, respectively.

Table 9 Probability $P\{g(X) < 0\}$

Failure probability	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X}) < 0\}$	7.007×10^{-7}	4.3581×10^{-7}	6.1754×10^{-4}	7.850×10^{-4}
N	1472	1514	102	10^6

Table 10 Probability at the tails of distribution

Failure probability	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X}) < 4.5 \times 10^7\}$	0.96798	0.97406	0.99927	0.99938
N	212	254	55	10^6

Table 11 Probability near the median

Failure probability	FORM	SORM	FOSPA	MCS
$P\{g(\mathbf{X}) < 2.48 \times 10^7\}$	0.1538	0.1406	0.4825	0.5038
N	93	135	43	10^6

where S is the material strength, D is the diameter of the shaft, F is the external force, T is the external torque, and L is the length of the shaft. The limit-state function represents the difference between the strength and the maximum stress.

The variable information is given in Table 8.

This problem belongs to case 1 where all of the random variables are tractable. The results of probability of failure as compared with MCS (10^6 simulations) are shown in Table 9. Referenced to MCS, FOSPA generates the most accurate solution with the least computational demand.

The preceding result indicates that FOSPA provides an accurate CDF estimate at the right tail of the distribution of the limit-state function. To illustrate the accuracy of FOSPA over the whole distribution range, the CDF of the limit-state function at the left tail and near the median are also calculated and given in Tables 10 and 11, respectively. From Tables 9 and 10, note that FOSPA is also superior to FORM and SORM at both tails in terms of accuracy and efficiency. Table 11 shows that FOSPA also produces a reasonably accurate CDF estimate around the median of the distribution whereas both FORM and SORM have very large errors. This example demonstrates that FOSPA is evenly accurate over the whole distribution and, therefore, beneficial for the generation of a complete distribution of a performance (limit-state function).

V. Discussion

In this section, we summarize the proposed FOSPA method with a detailed discussion on its accuracy and efficiency in comparison to FORM and SORM. Based on the discussion, recommendations for the selection of the reliability analysis methods under various circumstances will be provided in the next section.

Saddlepoint approximation is an accurate method for the estimation of CDF of a random variable if its CGF is known. The central idea of the proposed FOSPA is to approximate the CGF of a general limit-state function through linearization of limit-sate function. The linearization is conducted at the MLP, the point where the joint PDF of the random variables is at its maximum value for a given limit-state value. If a random variable does not have a closed-form CGF (intractable), it is transformed to another random variable with a tractable CGF before the linearization. In this paper, an intractable random variable is transformed to a standard normal variable. Note that other types of random variables with tractable CGF can also be used for the transformation. Once the limit-state function is in the form of a linear combination of tractable variables, the CGF of the limit-state function is easily obtained. The saddlepoint is the solution to the equation of the first derivative of the CGF equal to the limit-state value. Thereafter, the saddlepoint approximation solution is used to approximate the probability of failure or the reliability.

In contrast to FORM, which conducts linearization in the transformed standard normal space (which imposes nonlinear transformations), FOSPA linearizes the limit-state function in the original space of tractable random variables. As a consequence of the minimization of random variable transformation, FOSPA reduces the chance of an increase in the nonlinearity of the limit-state function. Therefore, the linearization of the limit-state function in FOSPA gives a more accurate approximation than that of FORM. Generally, FOSPA is more accurate than FORM, except in the following cases where they are equivalent: 1) all random variables have intractable CGF and they are transformed into standard normal variables, 2) all tractable random variables are normally distributed and all of the intractable random variables are transformed into standard normal variables, and 3) all random variables are normally distributed. In the aforementioned three cases, the MLP from FOSPA is identical to the MPP from FORM and, therefore, both methods have the same accuracy. In this sense, FORM is a special case of FOSPA.

It is generally recognized that SORM is more accurate than FORM, although there are a few exceptions; however, there is no such direct conclusion about the comparison between FOSPA and SORM in terms of their accuracy. One method is more accurate than the other depending on the problem under consideration. Generally speaking, when the limit-state function is less nonlinear in terms of original random variables or the nonnormal to normal transformation increases nonlinearity of the limit-state function significantly, FOSPA may have a higher accuracy than SORM.

The search of the MLP needs an iterative process where the limit-state function is evaluated repeatedly. Because a search for an MLP is a task similar to a search for an MPP, it is expected that FOSPA has at most the same order of magnitude of computational demand as that of FORM. In many cases, the search for the MLP is more efficient than the search for the MPP because the constraint function in the optimization model of the MLP is less nonlinear than that of the MPP. Note that the search of the saddlepoint does not consume any limit-state function evaluations. Because SORM needs the second-order derivative of a limit-sate function, it is generally much less efficient when the derivative is evaluated numerically.

Considering the same computational effort and higher accuracy of FOSPA compared to FORM, one may choose FOSPA for a reliability analysis. When higher accuracy is needed, one should also consider that, depending on the linearity of the limit-state and random distribution, SORM is not always better than FOSPA in terms of accuracy. The computational efficiency, accuracy, and implementation simplicity of the proposed method make it attractive for real-world reliability analysis. The authors have extensively applied the proposed method to various computationally intensive simulation models used in automotive engine design (see Hoffman et al.[34]). As indicated in example 2, FOSPA can also be used to generate accurate CDF associated with a range of limit-state values. This is accomplished by enumeration of the limit-state values, performance of linearization at all MLP associated with the limit state values, and calculation of the probability by the use of Eq. (16) or (17). By the use of this approach, FOSPA can accurately calculate the CDF at both tails, as well as around the median (or mean) of a distribution.

To further improve accuracy, the second-order saddlepoint approximation can be considered, and the key to the new development is how to identify the CGF of a second-order approximation of a limit-state function.

VI. Conclusions

In summary, the proposed FOSPA method for reliability analysis is an attractive alternative to the existing reliability analysis methods FORM and SORM. One may consider the following facts when selecting the reliability methods: FORM is a special case of FOSPA, and the latter is more accurate than the former with less or at most the same computational effort. If the limit-state function in the original space is less nonlinear than that of standard normal transformed space, FOSPA may be more accurate than SORM. SORM is less efficient, that is, it requires more function evaluations, than FOSPA and FORM.

Appendix A: FORM Is a Special Case of FOSPA

If none of the random variables is tractable, FORM produces the same result as FOSPA when standard normal transformation is employed. After a limit-state function $g(\boldsymbol{X})$ is approximated by a linear function in Eq. (30), the CGF of $q(\boldsymbol{U})$ is given by

$$K(t) = -\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{u^*} u_i^* t + \frac{1}{2}\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2 t^2 \tag{A1}$$

and its derivative is

$$K'(t) = -\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{u^*} u_i^* + \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2 t \tag{A2}$$

The saddlepoint is obtained from $K'(t) = 0$:

$$t = \sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{u^*} u_i^* \bigg/ \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2 \tag{A3}$$

The CGF at the saddlepoint becomes

$$K(t) = -\frac{1}{2}\left[\left(\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{u^*} u_i^*\right)^2 \bigg/ \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2\right] \tag{A4}$$

and its second-order derivative with respect to the saddlepoint is given by

$$K''(t) = \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2 \tag{A5}$$

Substitution of Eqs. (A4) and (A5) into Eqs. (18) and (19) yields

$$w = \sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{u^*} u_i^* \bigg/ \left[\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2\right]^{\frac{1}{2}} = -\beta \tag{A6}$$

$$\begin{aligned} v &= \left[\sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{u^*} u_i^* \bigg/ \sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2\right]\left\{2\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2\right\}^{\frac{1}{2}} \\ &= \sum_{i=1}^{n} \left.\frac{\partial g}{\partial u_i}\right|_{u^*} u_i^* \bigg/ \left[\sum_{i=1}^{n}\left(\left.\frac{\partial g}{\partial u_i}\right|_{u^*}\right)^2\right]^{\frac{1}{2}} = -\beta \end{aligned} \tag{A7}$$

respectively.

The combination of Eqs. (A6), (A7), and Eq. (16) results in

$$P\{g < 0\} = \Phi(-\beta) \tag{A8}$$

which is the same result as that of FORM.

Appendix B: Case 1 of Example 1

For FOSPA, the CGF of the limit-state function of example 1 is given by

$$K(t) = -n \ln(1-t) - \left(n + a\sqrt{n}\right)t \tag{B9}$$

and its derivatives are

$$K'(t) = n/(1-t) - \left(n + a\sqrt{n}\right) \tag{B10}$$

$$K''(t) = \left(n + a\sqrt{n}\right)^2 \big/ n \tag{B11}$$

respectively.

The solution of $K'(t) = 0$ produces the saddlepoint

$$t = a\sqrt{n} \big/ \left(n + a\sqrt{n}\right) > 0 \tag{B12}$$

Combining Eqs. (B9–B12), we obtain

$$w = \operatorname{sign}(t)\{2(ty - K(t)\}^{\frac{1}{2}} = 2\left\{n \ln\left[n \big/ \left(n + a\sqrt{n}\right)\right] + a\sqrt{n}\right\}^{\frac{1}{2}} \tag{B13}$$

$$v = t\{K''(t)\}^{\frac{1}{2}} = a \tag{B14}$$

and the probability of failure

$$p_f = \Phi\{w + (1/w)\ln(v/w)\} = \Phi\left\{2\left\{c + a\sqrt{n}\right\}^{\frac{1}{2}} + \left(1\big/2\left\{c + a\sqrt{n}\right\}^{\frac{1}{2}}\right)\ln\left(a\big/2\left\{c + a\sqrt{n}\right\}^{\frac{1}{2}}\right)\right\} \tag{B15}$$

where

$$c = n \ln\left[n\left(n + a\sqrt{n}\right)\right] \tag{B16}$$

For FORM the MPP $\boldsymbol{u}^* = \{u_i^*\}$, $i = 1, 2, \ldots, n$ is given by

$$u_i^* = -\Phi^{-1}\left\{\exp\left[-\left(n + a\sqrt{n}\right)\big/ n\right]\right\} \tag{B17}$$

and the reliability index is calculated by

$$\beta = \sqrt{n}u_i^* \tag{B18}$$

Then, the probability of failure is

$$p_f = \Phi(-\beta) = \Phi\left(-\sqrt{n}u_i^*\right) \tag{B19}$$

For SORM, the probability of failure is given by

$$p_f = \Phi^{-1}\left(-\sqrt{n}u_i^*\right)\left\{1 + u_i^*\left[\frac{u_i^*\Phi\left(-u_i^*\right) - \phi\left(-u_i^*\right)}{\Phi\left(-u_i^*\right)}\right]\right\}^{-(n-1)/2} \tag{B20}$$

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